Higher-order Algebraic Theories and Relative Monads

Nathanael Arkor   Dylan McDermott

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Outline

- Algebraic theories
- Second-order algebraic theories
- Higher-order algebraic theories
- Relative monads, monads, and theories

I. Algebraic theories

First-order operators

1. $$\frac{\Gamma \vdash a \qquad \Gamma \vdash b}{\Gamma \vdash a \times b}$$ Multiplication

2. $$\frac{\Gamma \vdash a}{\Gamma \vdash a^{-1}}$$ Inverses

3. $$\frac{\Gamma \vdash m : M \qquad \Gamma \vdash a : A}{\Gamma \vdash m * a : A}$$ Actions

Algebraic theories

identity-on-objects cartesian functor

$$\mathbb{L} \xrightarrow{\;L\;} \mathcal{L}$$

free cartesian category on a point

cartesian category

(Here, 'cartesian' means finite products.)

Algebraic theories

$$\mathbb{L} \xrightarrow{L} \mathcal{L}$$

The objects of $\mathcal{L}$ are given by X^n for X the generating object, and $n \in \mathbb{N}$.

A morphism $X^n \xrightarrow{\vec{t}} X^m$ represents an m-tuple of terms in n variables:

$$\langle x_1, \dots, x_n \vdash t_i \rangle_{1 \leq i \leq m}$$

Algebraic theories

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$$\langle x_1, \dots, x_n \vdash t_i \rangle_{1 \leq i \leq m}$$

n-ary operation $\langle t_i : X^n \longrightarrow X \rangle_i$

Algebraic theories

A map of algebraic theories is a commutative triangle

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathbb{L} \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

Algebraic theories and their maps form a category **Law**.

Monads and theories

There is a classic equivalence between algebraic theories and (strongly) finitary monads on the category of sets.

$$\mathrm{Law} \simeq \mathrm{Mnd}_f(\mathrm{Set}) = \mathrm{Mnd}_{sf}(\mathrm{Set})$$

Finitary = preserves filtered colimits

Strongly finitary = preserves sifted colimits (sifted-cocontinuous)

Universal algebra

$$\begin{tikzcd}
& \text{Equational logic} \arrow[dl, bend left] \arrow[dr, bend left] & \\
\text{Algebraic theories} \arrow[ur, bend left] \arrow[ur, phantom, "\simeq"] \arrow[rr, bend left] \arrow[rr, phantom, "\simeq"] & & \text{Monads on Set} \arrow[ul, bend left] \arrow[ul, phantom, "\simeq"] \arrow[ll, bend left]
\end{tikzcd}$$

II. SECOND-ORDER ALGEBRAIC THEORIES

[Fiore & Mahmoud, 2010]

Second-order operators

1. $$\frac{\Gamma, x \vdash f \qquad \Gamma \vdash x_0}{\Gamma \vdash \frac{df}{dx}(x_0)}$$

Differential operators (cf. Plotkin 2020)

2. $$\frac{\Gamma, x \vdash P}{\Gamma \vdash \exists x . P}$$

Logical quantifiers

3. $$\frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x . t}$$

λ-abstraction

Second-order operators

4. $$\frac{\Gamma \vdash t : A + B \qquad \Gamma, a : A \vdash u : C \qquad \Gamma, b : B \vdash v : C}{\Gamma \vdash \mathrm{case}(t,\ a.u,\ b.v) : C}$$

Coproducts, case-splitting

5. $$\frac{\Gamma, x : X \vdash f : X}{\Gamma \vdash \mathrm{fix}(f) : X}$$

Fixed points

operators
and equations
of the theory

$$\mathbb{L} \xrightarrow{L} \mathcal{L}$$

structural
operations

Second-order theory of equality

$\mathbb{L}_2$ is the free cartesian category with an exponentiable object (i.e. an object such that $(-)^X : \mathbb{L}_2 \to \mathbb{L}_2$ exists).

Objects of $\mathbb{L}_2$ are given by products

$$X^{X^{n_1}} \times \cdots \times X^{X^{n_k}}$$

with morphisms given by projection and evaluation.

Second-order theory of equality

$\mathbb{L}_2$ is the free cartesian category with an exponentiable object (i.e. an object such that $(-)^X : \mathbb{L}_2 \to \mathbb{L}_2$ exists).

Objects of $\mathbb{L}_2$ are given by products

$$X^{X^{n_1}} \times \cdots \times X^{X^{n_k}}$$

exponents are the objects of $\mathbb{L}$

with morphisms given by projection and evaluation.

Second-order algebraic theories

identity-on-objects cartesian functor, preserving the exponentiable object

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

free cartesian category on an exponentiable object

cartesian category with an exponentiable object

Second-order algebraic theories

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

A morphism $X^{X^{n_1}} \times \cdots \times X^{X^{n_k}} \xrightarrow{\vec{t}} X^{X^{m_1}} \times \cdots \times X^{X^{m_l}}$ in $\mathcal{L}$ represents an l-tuple of terms in k metavariables and m_i variables:

$$\langle (x^1_1, \ldots, x^1_{n_1})\, x_1, \ldots, (x^k_1, \ldots, x^k_{n_k})\, x_k, y_1, \ldots, y_{m_i} \vdash t \rangle_i$$

parameterised variable Ordinary variable

'Differentiate $f(x)$ with respect to x and evaluate at x_0.'

$$\partial(x, f(x), x_0)$$

represented by

$$X^X \times X \xrightarrow{\partial} X$$

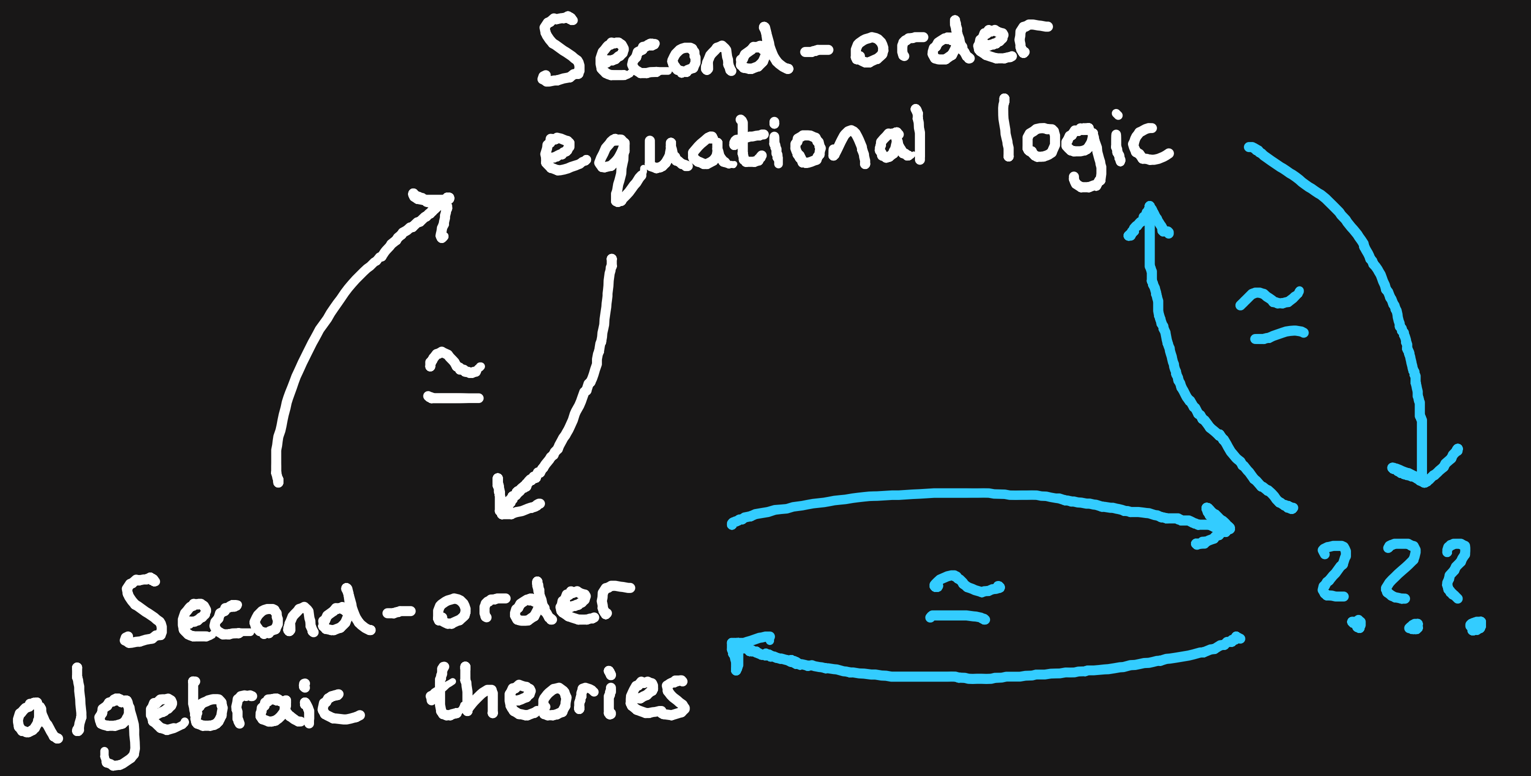
Second-order algebraic theories

A **map** of second-order algebraic theories is a commutative triangle

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathbb{L}_2 \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

Second-order algebraic theories and their maps form a category $\mathbf{Law}_2$.

Second-order universal algebra



III. HIGHER-ORDER ALGEBRAIC THEORIES

operators
and equations
of the theory

$$\mathbb{L} \xrightarrow{L} \mathcal{L}$$

structural
operations

Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an *n-tetrable object* (i.e. an object X such that

$$\underbrace{\overset{1}{1},\ \overset{2}{X},\ \overset{3}{X^X},\ \overset{4}{X^{X^X}},\ \overset{\dots}{\dots}}_{n}$$

is exponentiable).

Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an **n-tetrable object** (i.e. an object X such that

$$\underbrace{\overset{1}{1},\ \overset{2}{X},\ \overset{3}{X^X},\ \overset{4}{X^{X^X}},\ \overset{\dots}{\dots}}_{n}$$

is exponentiable).

Intuitively, morphisms in $\mathbb{L}_n$ represent operators taking operators as operands.

Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an **n-tetrable object** (i.e. an object X such that $1, X, X^X, X^{X^X}, \ldots$ is exponentiable).

We have:

$$\mathbb{L} = \mathbb{L}_1 \longleftrightarrow \mathbb{L}_2 \longleftrightarrow \cdots \longleftrightarrow \mathbb{L}_\omega$$

free cartesian category on a point

free cartesian-closed category on a point

Higher-order algebraic theories

identity-on-objects cartesian functor, preserving the n-tetrable object

$$\mathbb{L}_n \xrightarrow{L} \mathcal{L}$$

free cartesian category on an n-tetrable object

cartesian category with an n-tetrable object

Higher-order algebraic theories

A map of n^{th}-order algebraic theories is a commutative triangle

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathbb{L}_n \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

n^{th}-order algebraic theories and their maps form a category $\mathbf{Law}_n$.

Is there a monad correspondence for n^{th}-order algebraic theories?

The universal property of Law_n

Thm

Law_n is locally strongly finitely presentable.

$$\mathsf{Law}_n \simeq \mathrm{Cart}(\mathbb{L}_{n+1}, \mathrm{Set})$$
$$\simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$$

sifted cocompletion

free cartesian category on an $(n+1)$-tetrable point

The universal property of Law_n $(n=1)$

Thm

Law_1 is locally strongly finitely presentable.

$$\mathrm{Law} = \mathrm{Law}_1 \simeq \mathrm{Cart}(\mathbb{L}_2, \mathrm{Set})$$
$$\simeq \mathrm{Sind}(\mathbb{L}_2^{\circ})$$

sifted cocompletion

free cartesian category on an exponentiable object

The universal property of $\mathbb{L}\mathrm{aw}_n$

Thm

Law_n is locally strongly finitely presentable.

$$\mathrm{Law}_n \simeq \mathrm{Cart}(\mathbb{L}_{n+1}, \mathrm{Set})$$

$$\simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$$

sifted cocompletion

free cartesian category on an $(n+1)$-tetrable point

Hence also:

- Locally finitely presentable
- Cocomplete
- Complete

IV. Relative Monads

Higher-order algebraic theories $\overset{1}{\sim}$ Relative monads $\overset{2}{\sim}$ Monads

Monads

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=60, "T \circ T"] \arrow[rr, "T" description] \arrow[rr, bend right=60, "1"'] \arrow[rr, phantom, bend left=30, "\mu \Downarrow"] \arrow[rr, phantom, bend right=30, "\eta \Uparrow"] & & \mathcal{C}
\end{tikzcd}$$

subject to associativity and unitality laws

Relative monads

$$\mathcal{C} \xrightarrow[J]{} \mathcal{D}$$

Relative monads

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "J"'] \arrow[rr, bend left=60, "T"] \arrow[rr, phantom, bend left=25, "\eta \Uparrow"] & & \mathcal{D}
\end{tikzcd}$$

Relative monads

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "J"'] \arrow[rr, bend left=60, "T"] \arrow[rr, phantom, bend left=25, "\eta \Uparrow"] & & \mathcal{D} \end{tikzcd}$$

But what about multiplication?

Relative monads

A J-relative monad $(T, \eta, (-)^*)$ consists of

- a function $T : |\mathcal{C}'| \longrightarrow |\mathcal{C}|$
- a transformation $\eta_X : JX \longrightarrow TX$
- a transformation $(-)^*_{X,Y} : \mathcal{C}(JX, TY) \longrightarrow \mathcal{C}(TX, TY)$

satisfying unitality and associativity conditions.

Relative monads

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- a transformation $(-)^*_{X,Y} : \mathcal{C}(JX, TY) \longrightarrow \mathcal{C}(TX, TY)$

satisfying unitality and associativity conditions.

Prop: a monad is precisely an Id-relative monad.

We will consider the functor

$$\mathbb{L}_{n+1}^{\circ} \xhookrightarrow{\mathcal{L}} \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \simeq \mathrm{Law}_n$$

(intuitively a Yoneda embedding)

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$$\mathbb{L}_{n+1}^{\circ} \xrightarrow{\text{よ}} \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \simeq \mathrm{Law}_n$$

(intuitively a Yoneda embedding)

When $n=0$:

$$\mathbb{L}_1^{\circ} \xrightarrow{\text{よ}} \mathrm{Sind}(\mathbb{L}_1^{\circ})$$

free cocartesian category on a point

We will consider the functor

$$\mathbb{L}_{n+1}^{\circ} \xhookrightarrow{\;\;\text{よ}\;\;} \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \simeq \mathrm{Law}_n$$

(intuitively a Yoneda embedding)

When $n=0$:

$$\mathrm{FinSet} \simeq \mathbb{L}_1^{\circ} \xhookrightarrow{\;\;\text{よ}\;\;} \mathrm{Sind}(\mathbb{L}_1^{\circ}) \simeq \mathrm{Set}$$

free cocartesian category on a point

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}(\not{\!\!\smile}_{\amalg_{n+1}{}^{\circ}})$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\amalg_{n+1}{}^{\circ} \longleftrightarrow \mathrm{Law}_n)$-relative monads

Thm

$$\begin{tikzcd} & & \mathbb{L}_{n+1}^{\circ} \arrow[r, hook, "よ"] & \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \simeq \mathrm{Law}_n \arrow[d] \\ \mathrm{Law}_{n+1} & \simeq & & \mathrm{RMnd}_{+\text{-lin}}(よ_{\mathbb{L}_{n+1}^{\circ}}) \end{tikzcd}$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\mathbb{L}_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}(よ_{\mathbb{L}_{n+1}^{\circ}})$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\mathbb{L}_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

When $n=0$, this says that algebraic theories are equivalent to $(\mathrm{FinSet} \hookrightarrow \mathrm{Set})$-relative monads.

Thm

technical condition, imposing invertibility of a canonical strength

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}(\mathcal{L}_{\amalg_{n+1}^{\circ}})$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\amalg_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}(よ_{\amalg_{n+1}^{\circ}})$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\amalg_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

(But what about ordinary monads?)

$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$

Theory | Relative monad | Monad

$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$

$\mathbb{L}_1^\circ \xrightarrow[T_L]{} \mathrm{Sind}(\mathbb{L}_1^\circ)$

Theory	Relative monad	Monad

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[dr, dashed, "\mathrm{Lan}_{よ} T_L"] & \\
\mathbb{L}_1^\circ \arrow[ur, "よ"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[dr, dashed, "\mathrm{Lan}_{\mathcal{L}} T_L"] & \\
\mathbb{L}_1^\circ \arrow[ur, "\mathcal{L}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$T_L' = \mathrm{Lan}_{\mathcal{L}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above]
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] \\
\mathbb{L}_1^\circ \arrow[ur, "\text{よ}"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above] \simeq \mathrm{Set}
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[dr, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] & \\
\mathbb{L}_1^\circ \arrow[ur, "\text{よ}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above] \simeq \mathrm{Set}
\end{tikzcd}$$

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

Theory | Relative monad | Monad

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\mathbb{L}_2^\circ \xrightarrow[T_L]{} \mathrm{Sind}(\mathbb{L}_2^\circ)$$

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] \\
\mathbb{L}_1^\circ \arrow[ur, "\text{よ}"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$T_L' = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above] \simeq \mathrm{Set}
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_2^\circ) \arrow[dr, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] & \\
\mathbb{L}_2^\circ \arrow[ur, "\text{よ}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_2^\circ)
\end{tikzcd}$$

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[dr, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] & \\
\mathbb{L}_1^\circ \arrow[ur, "\text{よ}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above, "T'_L"] \simeq \mathrm{Set}
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
 & \mathrm{Sind}(\mathbb{L}_2^{\circ}) \arrow[dr, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] & \\
\mathbb{L}_2^{\circ} \arrow[ur, "\text{よ}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_2^{\circ})
\end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_2^{\circ}) \arrow[loop above]
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$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

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 & \mathrm{Sind}(\mathbb{L}_1^{\circ}) \arrow[dr, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] & \\
\mathbb{L}_1^{\circ} \arrow[ur, "\text{よ}"] \arrow[rr, "T_L"'] & & \mathrm{Sind}(\mathbb{L}_1^{\circ})
\end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{\text{よ}} T_L$$

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_1^{\circ}) \arrow[loop above] \simeq \mathrm{Set}
\end{tikzcd}$$

Theory | Relative monad | Monad

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd} & \mathrm{Sind}(\mathbb{L}_2^\circ) \arrow[d, dashed, "\mathrm{Lan}_{よ} T_L"] \\ \mathbb{L}_2^\circ \arrow[ur, "よ"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_2^\circ) \end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{よ} T_L$$

$$\begin{tikzcd} \mathrm{Sind}(\mathbb{L}_2^\circ) \arrow[loop above, "T'_L"] \end{tikzcd} \simeq \mathrm{Law}$$

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd} & \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[d, dashed, "\mathrm{Lan}_{よ} T_L"] \\ \mathbb{L}_1^\circ \arrow[ur, "よ"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_1^\circ) \end{tikzcd}$$

$$T'_L = \mathrm{Lan}_{よ} T_L$$

$$\begin{tikzcd} \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above, "T'_L"] \end{tikzcd} \simeq \mathrm{Set}$$

Theory | Relative monad | Monad

$$\vdots$$

$$\mathbb{L}_3 \xrightarrow{L} \mathcal{L}$$

$$\mathbb{L}_3^\circ \xrightarrow[T_L]{} \mathrm{Sind}(\mathbb{L}_3^\circ)$$

$$\begin{tikzcd} \mathrm{Sind}(\mathbb{L}_3^\circ) \arrow[loop above, "T'_L = \mathrm{Lan}_{\text{よ}} T_L"] \end{tikzcd} \simeq \mathrm{Law}_2$$

$$\mathbb{L}_2 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
& \mathrm{Sind}(\mathbb{L}_2^\circ) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] \\
\mathbb{L}_2^\circ \arrow[ur, "\text{よ}"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_2^\circ)
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sind}(\mathbb{L}_2^\circ) \arrow[loop above, "T'_L = \mathrm{Lan}_{\text{よ}} T_L"] \end{tikzcd} \simeq \mathrm{Law}$$

$$\mathbb{L}_1 \xrightarrow{L} \mathcal{L}$$

$$\begin{tikzcd}
& \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}} T_L"] \\
\mathbb{L}_1^\circ \arrow[ur, "\text{よ}"] \arrow[r, "T_L"'] & \mathrm{Sind}(\mathbb{L}_1^\circ)
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Sind}(\mathbb{L}_1^\circ) \arrow[loop above, "T'_L = \mathrm{Lan}_{\text{よ}} T_L"] \end{tikzcd} \simeq \mathrm{Set}$$

Theory | Relative monad | Monad

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}\left(\text{よ}_{\mathbb{L}_{n+1}^{\circ}}\right)$$

$(n+1)^{th}$-order algebraic theories

$+$-linear $(\mathbb{L}_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

$$\simeq \mathrm{Mnd}_{+\text{-lin},\, \mathrm{sf}}(\mathrm{Law}_n)$$

sifted-cocontinuous $+$-linear monads on Law_n

Idea

Variable-binding structure is algebraic over algebraic structure.

Coreflections

Thm

There is a coreflection of categories:

inclusion of presentations

$$\begin{tikzcd} \mathrm{Law}_n \arrow[r, hook, shift left=1.5ex, "\ulcorner - \urcorner"] \arrow[r, phantom, "\perp"] & \mathrm{Law}_{n+1} \arrow[l, shift left=1.5ex, "\llcorner - \lrcorner"] \end{tikzcd}$$

discard $(n+1)^{th}$- order terms

Coreflections

There is a chain of coreflections,

$$\begin{tikzcd} \mathrm{Law}_1 \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Law}_2 \arrow[l, shift left] \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, shift left] \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Law}_\omega \arrow[l, shift left] \end{tikzcd}$$

allowing us to freely extend or restrict the order of a higher-order algebraic theory.

Prop.

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory.

The corresponding monad is given by

$$T_L(X) \cong {}_{L}L + \ulcorner X \urcorner$$

Prop.

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory. The corresponding monad is given by

$$T_L(X) \cong \llcorner L + \ulcorner X \urcorner \lrcorner$$

When $n=0$, this says that T_L takes a set of constants, freely adds them to L, then extracts the new constants formed from those in X under the operations of L.

Summary

- Higher-order algebraic theories generalise algebraic theories by (higher-order) variable binding operators.
- There are coreflections
$$\begin{tikzcd} \mathrm{Law}_n \arrow[r, hook, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Law}_{n+1} \arrow[l, shift left=1.5] \end{tikzcd}.$$
- $\mathrm{Law}_n \simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$
- $\mathrm{Law}_{n+1} \simeq \mathrm{Mnd}_{sf,+\text{-}lin}(\mathrm{Law}_n)$

Algebras

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory, and let $T_L : \mathrm{Law}_n \longrightarrow \mathrm{Law}_n$ be the corresponding monad.

$$T_L\text{-}\mathrm{Alg} \simeq \mathrm{Cart}(\mathcal{L}, \mathrm{Set})$$